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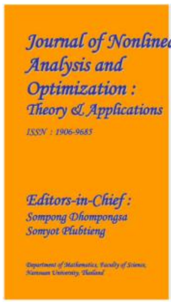

(n, m) NORMAL AND (n, m) QUASI-NORMAL OPERATORS IN MINKOWSKI SPACE M

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Abstract
In this paper, we have discuss about the some theorems and proposition of (n, m) normal and (n, m) quasi-normal operators in Minkowski space.

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1. Introduction

Throughout we shall deal with $C^{n\times n}$, the space of $n\times n$ complex matrices. Let C^n be the space of complex n-tuples, we shall index the components of a complex vector in C^n from 0 to n-1, that is $u = (u_0, u_1, u_2, ..., u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, ..., -u_{n-1})$. Clearly the Minkowski metrix matrix

$$G=\begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \qquad (1)$$

$G = G^*$ and $G^2 = I_n$. In[9], Minkowski inner product on C^n is defined by $(u, v) = [u, Gv]$, where $[.,.]$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called the Minkowski space and denoted as M. For $A \in C^{n\times n}, x, y \in C^n$ by using (1),
$(Ax, y) = [Ax, Gy] = [x, A^*Gy]$
$= [x, G(GA^*G)y] = [x, GA^\sim y] = (x, A^\sim y)$.
where $A^\sim = GA^*G$. The matrix $A^\sim$ is called the Minkowski adjoint of A in M. Naturally, we call a matrix $A \in C^{n\times n}$ m-symmetric in M if $A = A^\sim$. For $A \in C^{n\times n}$, let A^*, $A^\sim$, A^{m}, $A^\dagger$, $R(A)$ and $N(A)$ denote the conjugate transpose, Minkowski adjoint, Minkowski inverse, Moore-Penrose inverse, range space and null space of a matrix A respectively. I_n denote the identity matrix of order $n\times n$.
Generalized inverses of matrices have important roles in theoretical and numerical methods of linear algebra. The most significant fact is that we can use generalized inverse of matrices, in the case when ordinary inverses do not exists, in order to solve some matrix equations. Similar reasoning can be applied to linear (bounded or unbounded) operators on Banach and Hilbert spaces.
n-power quasi-normal operators were defined and discussed by mecheri [7] and ould Ahmed mahamoud sid Ahmed [8]. Alzuraiqi [3] and jibril [6] introduced the n-power normal operators. Moreover Al-Loz [2] discussed about the concepts of (n, m)-normal powers operators on Hilbert space. Several characteristics of (n, m) normal and (n, m) qusai-normal operators will be examined in this paper. We will see some related theorems and proposition are given below.

2. Preliminaries

Definition 2.1. An operator E is said to be normal in M if $E^\sim E = E E^\sim$
Definition 2.2. An operator E is (n, m) normal operator in M, if $E^n(E^{m\sim}) = (E^{m\sim})E^n$ here (n, m) are positive integer.

Definition 2.3. An operator E is called n-normal in M if E^n is a normal operator in M This is equivalent to $E^{\sim} E^n = E^n E^{\sim}$.

Definition 2.4. An operator E is called n-quasi-normal operator in M, if $(E^{\sim} E^n)E = E(E^n E^{\sim})$.

3. (n, m) normal operator in Minkowski space

In this section some basic properties of (n, m) normal operator in Minkowski space are derived.

Proposition 3.1. If E is operator in Minkowski space in M and E be (n, m) normal operator in Minkowski space in M, then E^{nm} is (n,m) normal operator in Minkowski space in M.

Proof:

Given

E is (n, m) normal operator in Minkowski space in M

So $E^n(E^{m\sim}) = (E^{m\sim})E^n$

We know that $(E^k)^{\sim} = (E^{\sim})^k$ for each non-negative integer.

To prove E^{nm} normal operator in Minkowski space M

$$\begin{aligned}
(E^{nm})(E^{nm})^{\sim} &= (E^n)^m ((E^m)^n)^{\sim} \\
&= \underbrace{(E^n E^n \ldots\ldots E^n)}_{\text{m-times}} \underbrace{(E^m E^m \ldots\ldots E^m)^{\sim}}_{\text{n-times}} \\
&= E^n E^n \ldots\ldots E^n (E^m)^{\sim} (E^m)^{\sim} \ldots\ldots (E^m)^{\sim} \\
&= E^n E^n \ldots\ldots (E^m)^{\sim} E^n \ldots\ldots (E^m)^{\sim} \\
&= (E^m)^{\sim} (E^m)^{\sim} \ldots\ldots (E^m)^{\sim} E^n E^n \ldots\ldots E^n \\
&= \underbrace{(E^m E^m \ldots\ldots E^m)^{\sim}}_{\text{n-times}} \underbrace{(E^n E^n \ldots\ldots E^n)}_{\text{m-times}} \\
&= ((E^m)^n)^{\sim} (E^n)^m \\
&= ((E^n)^m)^{\sim} (E^n)^m \\
&= (E^{nm})^{\sim} (E^{nm})
\end{aligned}$$

$$(E^{nm})(E^{nm})^{\sim} = (E^{nm})^{\sim} (E^{nm})$$

Therefore (E^{nm}) is normal operator in Minkowski space M.

Proposition 3.2. E be (n, m) normal operator in Minkowski space M iff E be (m, n) normal operator in Minkowski space M.

Proof:

Given E be (n, m) normal operator in Minkowski space M.

So $E^n (E^m)^{\sim} = (E^m)^{\sim} E^n$

$$\begin{aligned}
\text{Take } E^m (E^n)^{\sim} &= ((E^m (E^n)^{\sim})^{\sim})^{\sim} \\
&= (E^n (E^m)^{\sim})^{\sim} \\
&= ((E^m)^{\sim} E^n)^{\sim} \\
&= (E^n)^{\sim} E^m
\end{aligned}$$

$$E^m (E^n)^{\sim} = (E^n)^{\sim} E^m$$

Therefore E be (n, m) normal operator in Minkowski space M.

Proposition 3.3. E be (n, m) normal operator in Minkowski space M. Then

i) $E^{\sim}$is (n, m) normal operator in M..

ii) E^{-1} exists then E^{-1} is (n, m) normal operator in M.

iii) E is unitary equivalent to F then, E is (n, m) normal operator in M.

Proof:

i) Given E is (n, m) normal operator in M.

So, We have $E^n (E^m)^\sim = (E^m)^\sim E^n$
Replace E by $E^\sim$

$(E^\sim)^n ((E^\sim)^\sim)^m = (E^\sim)^n E^m$

$= (E^m)(E^\sim)^n$ [by using 3.2 proposition]

$= ((E^\sim)^\sim)^m (E^\sim)^n$

Therefore $(E^\sim)^n ((E^\sim)^\sim)^m = ((E^\sim)^\sim)^m (E^\sim)^n$

ii) $(E^{-1})^n ((E^{-1})^\sim)^m = ((E^\sim)^m E^n)^{-1}$

$= ((E^n) (E^m)^\sim)^{-1}$ [Since E is (n, m) normal operator]

$= ((E^m)^\sim)^{-1} (E^n)^{-1}$

$= ((E^{-1})^\sim)^m (E^{-1})^n$

$(E^{-1})^n ((E^{-1})^\sim)^m = ((E^{-1})^\sim)^m (E^{-1})^n$

Therefore E^{-1} is (n, m) normal operator in Minkowski space M .
iii) Since E is unitary equivalent to F then E = IF $I^\sim$and (IF $I^\sim)^n$ = I$F^n I^\sim$
To prove I be (n,m) normal operator in Minkowski space M.

Take $E^n (E^m)^\sim = (\mathrm{IF}\, I^\sim)^n ((\mathrm{IF}\, I^\sim)^m)^\sim$

$= \mathrm{I}F^n I^\sim ((\mathrm{I}F^m I^\sim)^\sim$

$= \mathrm{I}F^n I^\sim \mathrm{I}(F^m)^\sim I^\sim$

$= \mathrm{I}F^n (F^m)^\sim I^\sim$

$= \mathrm{I}(F^m)^\sim F^n I^\sim$

$= \mathrm{I}(F^m)^\sim I^\sim \mathrm{I}F^n I^\sim$

$= ((F^m I^\sim))^\sim \mathrm{I}F^n I^\sim$

$= (\mathrm{I}(F^m I^\sim))^\sim \mathrm{I} F^n I^\sim$

$= ((\mathrm{I\,F}I^\sim)^m)^\sim (\mathrm{IF}I^\sim)^n$

$= (E^m)^\sim E^n$

Therefore E be (n,m) normal operator in MInkowski space M.

Proposition 3.4. If F is (k,m) normal and $(k+1,m)$ normal operator in Minkowski space M here k,m are positive integer then F be $(k+2,m)$ normal operator in Minkowski space Mand by induction hypothesis F is (n,m) normal operator in Minkowski space M for all (n, m).
Proof:
Given F be (k,m) normal operator in Minkowski space M and F be $(k+1,m)$ normal operator in Minkowski space M
So we have

$F^k (F^m)^\sim = (F^m)^\sim F^k$ and $F^{k+1} (F^m)^\sim = (F^m)^\sim F^{k+1}$

Similarly $F^{k+2} (F^m)^\sim = \mathrm{F}\, F^{k+1} (F^m)^\sim$

$= \mathrm{F}\, (F^m)^\sim F^{k+1}$

$= \mathrm{F}\, (F^m)^\sim F^k\, \mathrm{F}$

$= \mathrm{F}\, F^k (F^m)^\sim\, \mathrm{F}$

$= F^{k+1} (F^m)^\sim\, \mathrm{F}$

$= (F^m)^\sim F^{k+2}$

$F^{k+2} (F^m)^\sim = (F^m)^\sim F^{k+2}$

Therefore by induction hypothesis F be $(k+2,m)$ normal operator in Minkowski space M
$\Longrightarrow F$ be (n,m) normal operator in Minkowski space M for all n, m.

Theorem 3.5. Let E commutes with F. If E and F are (n, m) normal operators in Minkowski space M then EF be (n,m) normal operators in Minkowski space M.

Proof:

Let E commutes with F and $(EF)^n = E^n F^n$

Additionally E commutes with $F^{\sim}$ and F commutes with $E^{\sim}$

Take

$$((EF)^n (EF)^m)^{\sim} = E^n F^n \,(E^m F^m)^{\sim}$$
$$= E^n F^n \,(F^m)^{\sim} (E^m)^{\sim}$$
$$= E^n (F^m)^{\sim} F^n \,(E^m)^{\sim}$$
$$= (F^m)^{\sim} E^n \,(E^m)^{\sim} F^n$$
$$= (F^m)^{\sim} (E^m)^{\sim} E^n F^n$$
$$= ((EF)^m)^{\sim} (EF)^n$$

Hence EF be (n,m) normal operators in Minkowski space M.

Proposition 3.6. If F be operator in Minkowski space M, let A= $F^n + (F^m)^{\sim}$, B= F^n - $(F^m)^{\sim}$ and D = $F^n (F^m)^{\sim}$. F be (n, m) normal operator, then if D commutes with A and B in M.

Proof:

Given F be (n,m) normal operator.

$$\text{DA} = F^n (F^m)^{\sim} F^n + (F^m)^{\sim}$$
$$= F^n (F^m)^{\sim} F^n + F^n (F^m)^{\sim} (F^m)^{\sim}$$
$$= F^n F^n (F^m)^{\sim} + (F^m)^{\sim} F^n (F^m)^{\sim}$$
$$= (F^n + (F^m)^{\sim}) F^n (F^m)^{\sim}$$
$$= \text{AD}$$

Therefore DA= AD

Similarly DB = BD

4. (n,m) Quasi-normal operator in Minkowski M

Theorem 4.1. Let F be (n,m) quasinormal operator M.then so are

i) γ for any scalar

ii) If E is operator in Minkowskispace M which is unitarily equivalent to F in M then E is (n,m) quasinormal operator M.

iii) The restriction F/A of F to any closed subspace A of H that redues F. Then F/A is (n,m) quasinormal operator M.

Proof:

Given F is (n,m) Quasinormal operator in Minkowski space M.

So $F^n (F^{\sim m} F) = (F^{\sim m} F) F^n$

Here n, m are non-negative integers.

$$i)\ (\gamma F)^n ((\gamma F)^{\sim m} \gamma F)$$
$$= \gamma^n F^n (\bar{\gamma}^m F^{\sim m} \gamma F)$$
$$= \gamma^n \bar{\gamma}^m \gamma F^n (F^{\sim m} F)$$
$$= \gamma^n \bar{\gamma}^m \gamma (F^{\sim m} F) F^n$$
$$= (\bar{\gamma}^m F^{\sim m} F \gamma F) \gamma^n F^n$$
$$= ((\gamma F)^{\sim m} \gamma F) (\gamma F)^n$$

Hence γ F is (n,m) quasinormal operator in Minkowski space M

ii) Let E= IF$I^{\sim}$ here I is a unitary operator

Take E^{n} $(E^{\sim m}\ E)$ = (IF$I^{\sim}$)^{n} ((IF$I^{\sim}$)$^{\sim m}$ (IF$I^{\sim}$))

= I $F^{n}I^{\sim}$ (I$F^{\sim m}$ $I^{\sim}$) IF$I^{\sim}$

= I $F^{n}(F^{\sim m}\ F)\ I^{\sim}$

= I $(F^{\sim m}\ F)\ F^{n}\ I^{\sim}$

= (I$F^{\sim m}$ $I^{\sim}I$F $I^{\sim}I$) F^{n} $I^{\sim}$

= ($I^{\sim}$ FI)$^{\sim m}$ (IF$I^{\sim}$) (IF$I^{\sim}$)^{n}

= $(E^{\sim m}\ E)\ E^{n}$

E^{n} $(E^{\sim m}\ E) = E^{\sim m}\ E)\ E^{n}$

Therefore E is (n, m) quasi-normal operator in Minkowski space M.

iii) $(F/A)^{n}$ $((F/A)^{m\sim}$ $(F/A))$

= (F^{n}/A)(($F^{m\sim}$/A)(F/A))

= (F^{n}($F^{m\sim}$F)/ A)

= (($F^{m\sim}$F) F^{n}/A)

= (($F^{m\sim}$/A)(F/A)) (F^{n}/A)

= ((F/A)$^{m\sim}$(F/A)) (F/A)^{n}

$(F/A)^{n}$ $((F/A)^{m\sim}$ $(F/A))$ = ((F/A)$^{m\sim}$(F/A)) (F/A)^{n}

Therefore F/A is (n, m) quasi-normal operator in Minkowski space M.

Theorem 4. 2 If E and F are (n $,m$) quasi-normal operator in Minkowski space M such that $EF = FE = E^{\sim}F =$ F$E = 0$ then $E+F$ is (n,m) quasi-normal operator in Minkowski space M.

Proof:

$(E+F)^{n}(E+F)^{\sim m}(E+F)$

$= (E^{n}+F^{n})\ (E^{\sim m}+F^{\sim m})(E+F)$

$= (E^{n}+F^{n})(E^{\sim m}E+F^{\sim m}F+E^{\sim m}F+F^{\sim m}E)$

$$= \left(E^{n}E^{\sim m}E+F^{n}F^{\sim m}F+E^{n}E^{\sim m}F+F^{n}E^{\sim m}F+E^{n}F^{\sim m}E+F^{n}F^{\sim m}E + (E^{n}F^{\sim m}E+F^{n}E^{\sim m}E)\right)$$

$=E^{nE^{\sim m}}E+F^{n}F^{\sim m}F$

$= (E^{\sim m}E^{n}E+F^{\sim m}F^{n}F)$ [Since E and F are quasi-normal operators]

$=(E^{\sim m}E^{n}\mathrm{E}+F^{\sim m}F^{n}\mathrm{F})$

$= (E^{\sim m}EE^{n}+F^{\sim m}FF^{n})$

$= (E^{\sim m}E^{n}+E^{\sim m}F^{n})\mathrm{E}+(F^{\sim m}E^{n}+F^{\sim m}F^{n})\mathrm{F}$

$=(E^{\sim m}\mathrm{E}+F^{\sim m}\mathrm{F})\ (E^{n}+F^{n})$

$=(E^{\sim m}\mathrm{E}+F^{\sim m}\mathrm{F}+E^{\sim m}\mathrm{F}+F^{\sim m}\mathrm{E})\ (E^{n}+F^{n})$

$=[(E^{\sim m}+F^{\sim m})\mathrm{E}+(E^{\sim m}+F^{\sim m})\mathrm{F}]\ (E^{n}+F^{n})$

$= (E^{\sim m}+F^{\sim m})\ (\mathrm{E+F})\ (E^{n}+F^{n})$

$= (E+F)^{\sim m}(E+F)(E^{n}+F^{n})$

$((E+F)^{n}(E+F)^{\sim m}(E+F) = (E+F)^{\sim m}(E+F)(E^{n}+F^{n})$

Hence $E+F$ is (n,m) quasinormal operator in Minkowski space M.

Theorem 4.3 If $E_1, E_2, \cdots, E_r$ are (n,m) Quasi normal operator in Minkowski space M. Then $(E_1 \oplus E_2 \oplus \ldots \oplus E_r)$ and $(E_1 \otimes E_2 \otimes \ldots \otimes E_r)$ are (n,m) Quasi normal operator in Minkowski space M.

proof:

(i) $(E_1 \oplus E_2 \oplus \cdots \oplus E_r)^n\big((E_1 \oplus E_2 \oplus \cdots \oplus E_r)^{\sim m}\ (E_1 \oplus E_2 \oplus \cdots \oplus E_r)\big)$

$= (E_1^n \oplus E_2^n \oplus \cdots \oplus E_r^n)(E_1^{\sim m} \oplus E_2^{\sim m} \oplus \cdots \oplus E_r^{\sim m})(E_1 \oplus E_2 \oplus \cdots \oplus E_r)$

$= E_1^n(E_1^{\sim m}E_1) \oplus \ldots\ldots\ldots\ldots \oplus E_r^n(E_r^{\sim m}E_r)$

$= (E_1^{\sim m}E_1)E_1^n \oplus \ldots\ldots\ldots\ldots \oplus (E_r^{\sim m}E_r)E_r^n$

$= \big((E_1^{\sim m} \oplus E_2^{\sim m} \oplus \cdots \oplus E_r^{\sim m})(E_1 \oplus E_2 \oplus \cdots \oplus E_r)\big)(E_1^n \oplus E_2^n \oplus \cdots \oplus E_r^n)$

$\big((E_1 \oplus E_2 \oplus \cdots \oplus E_r)^{\sim m}(E_1 \oplus E_2 \oplus \cdots \oplus E_r)\big)(E_1 \oplus E_2 \oplus \cdots \oplus E_r)^n$

$(E_1 \oplus E_2 \oplus \cdots \oplus E_r)^n\big((E_1 \oplus E_2 \oplus \cdots \oplus E_r)^{\sim m}(E_1 \oplus E_2 \oplus \cdots \oplus E_r)\big) = \big((E_1 \oplus E_2 \oplus \cdots \oplus E_r)^{\sim m}(E_1 \oplus E_2 \oplus \cdots \oplus E_r)\big)(E_1 \oplus E_2 \oplus \cdots \oplus E_r)^n$

Therefore $(E_1 \oplus E_2 \oplus \ldots \oplus E_r)$are (n,m) Quasi normal operator in Minkowski space M.

(ii) $(E_1 \otimes E_2 \otimes \cdots \otimes E_r)^n\big((E_1 \otimes E_2 \otimes \cdots \otimes E_r)^{\sim m}(E_1 \otimes E_2 \otimes \cdots \otimes E_r)\big)(x_1 \otimes \ldots \otimes x_r)$

$= (E_1^n \otimes E_2^n \otimes \cdots \otimes E_r^n)(E_1^{\sim m} \otimes E_2^{\sim m} \otimes \cdots \otimes E_r^{\sim m})(E_1 \otimes E_2 \otimes \cdots \otimes E_r)(x_1 \otimes \ldots \otimes x_r)$

$= E_1^n(E_1^{\sim m}E_1) \otimes \ldots\ldots \otimes E_r^n(E_r^{\sim m}E_r)(x_1 \otimes \ldots \otimes x_r)$

$= (E_1^{\sim m}E_1)E_1^n \otimes \ldots\ldots \otimes (E_r^{\sim m}E_r)E_r^n\ (x_1 \otimes \ldots \otimes x_r)$

$= \big((E_1^{\sim m} \otimes E_2^{\sim m} \otimes \cdots \otimes E_r^{\sim m})(E_1 \otimes E_2 \otimes \cdots \otimes E_r)\big)(E_1^n \otimes E_2^n \otimes \cdots \otimes E_r^n)\ (x_1 \otimes \ldots \otimes x_r)$

$=\big((E_1 \otimes E_2 \otimes \cdots \otimes E_r)^{\sim m}(E_1 \otimes E_2 \otimes \cdots \otimes E_r)\big)(E_1 \otimes E_2 \otimes \cdots \otimes E_r)^n\ (x_1 \otimes \ldots \otimes x_r)$

Hence $(E_1 \otimes E_2 \otimes \cdots \otimes E_r)^n\big((E_1 \otimes E_2 \otimes \cdots \otimes E_r)^{\sim m}(E_1 \otimes E_2 \otimes \cdots \otimes E_r)\big)(x_1 \otimes \ldots \otimes x_r)$

$= \big((E_1 \otimes E_2 \otimes \cdots \otimes E_r)^{\sim m}(E_1 \otimes E_2 \otimes \cdots \otimes E_r)\big)(E_1 \otimes E_2 \otimes \cdots \otimes E_r)^n\ (x_1 \otimes \ldots \otimes x_r)$.

Therefore $(E_1 \otimes E_2 \otimes \ldots \otimes E_r)\ are\ (n,m)$ Quasi-normal operator in Minkowski space M.

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